Asymptotic equidistribution for partition statistics and topological invariants

joint work with William Craig and Joshua Males

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Motivation

A *partition* λ of a positive integer n is a list of non-increasing positive integers, say $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, that satisfies $|\lambda| := \lambda_1 + \cdots + \lambda_m = n$.

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Thus we have $p(4) = 5$.

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To say that equidistribution holds is to say that

$$c(a, b; n) \sim \frac{1}{b} c(n)$$

as $n \to \infty$.

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$$h(k, j) := (\lambda_k - k) + (\lambda'_j - j) + 1,$$

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Bringmann–Craig–Males–Ono:
On arithmetic progressions modulo odd primes t-hooks are not asymptotically equdistributed. The Betti numbers of two specific Hilbert schemes are asymptotically equdistributed.

Wright's Circle Method

Hardy–Ramanujan, 1918

$$p(n) \sim \frac{1}{4\sqrt{3n}} \cdot e^{\pi\sqrt{\frac{2n}{3}}}, \qquad \text{as } n \to \infty.$$

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$$\mathcal{A}(\tau) := \sum_{n\geq 0} a(n)q^n \quad \longrightarrow \quad a(n) = \frac{1}{2\pi i}\int_C \frac{\mathcal{A}(q)}{q^{n+1}}dq,$$

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Following Wright and the work of Ngo–Rhoades,
Bringmann–Craig–Males–Ono proved the following variant of Wright's Circle Method.

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Let $M > 0$ be a fixed constant and $z = x + iy \in \mathbb{C}$, with $x > 0$ and $|y| < \pi$.

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(i) As $z \to 0$ in the bounded cone $|y| \leq Mx$ (major arc), we have

$$F(e^{-z}) = z^B e^{\frac{A}{z}} \left(\alpha_0 + O_M\left(|z|\right)\right),$$

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(ii) As $z \to 0$ in the bounded cone $Mx \leq |y| < \pi$ (minor arc), we have

$$|F(e^{-z})| \ll_M e^{\frac{1}{\mathrm{Re}(z)}(A-\kappa)},$$

for some $\kappa \in \mathbb{R}^+$.

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Suppose that $F(q)$ is analytic for $q = e^{-z}$ where $z = x + iy \in \mathbb{C}$ satisfies $x > 0$ and $|y| < \pi$,

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where $p_0 = \alpha_0 \dfrac{\sqrt{A}^{B+\frac{1}{2}}}{2\sqrt{\pi}}$.

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for some generating functions $H(\zeta; q)$, with

$$H(q) := H(1; q) = \sum_{n \geq 0} c(n) q^n.$$

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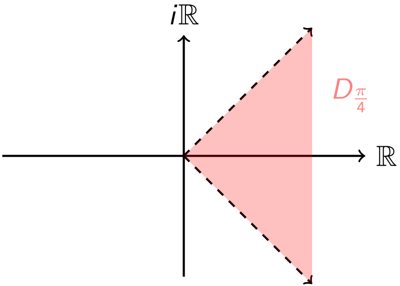

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(3) As $n \to \infty$, we have

$$c(n) \sim \frac{1}{2\pi i} \int_{\widetilde{C}} \frac{H(1; q)}{q^{n+1}} dq.$$

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C.–Craig–Males, 2021

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In particular, if $H(1; q)$ and $H(\zeta; q)$ satisfy the conditions of BCMO we have that

$$c(a, b; n) \sim \frac{1}{b} c(n) \sim \frac{1}{b} n^{\frac{1}{4}(-2B-3)} e^{2\sqrt{An}} \left(p_0 + O\left(n^{-\frac{1}{2}} \right) \right)$$

as $n \to \infty$.

Idea of the proof

1. Use Cauchy's theorem and the decomposition of $H(a, b; q)$ to obtain

$$c(a, b; n) = \frac{1}{b}\left[\frac{1}{2\pi i}\int_C \frac{\sum_{j=0}^{b-1}\zeta_b^{-aj} H(\zeta_b^j; q)}{q^{n+1}} dq\right].$$

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2. Break down the integral over C into the components $\widetilde{C}$ and $C\backslash\widetilde{C}$ and look at each of them seperately.
3. Along $C\backslash\widetilde{C}$ we have by conditions (1) and (3) that as $n \to \infty$

$$\frac{1}{2\pi i}\int_{C\backslash\widetilde{C}} \frac{\sum_{j=0}^{b-1}\zeta_b^{-aj}H(\zeta_b^j; q)}{q^{n+1}}dq = o\left(\frac{1}{2\pi i}\int_{\widetilde{C}}\frac{H(1; q)}{q^{n+1}}dq\right).$$

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4. On $\widetilde{C}$ we obtain with (2) and as $n \to \infty$ that

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5. The first claim follows by combining the estimates along $\widetilde{C}$ and $C \setminus \widetilde{C}$.
6. If we assume $H(1; q)$ and $H(\zeta_b^j; q)$ satisfy the hypotheses of BCMO, then (1) – (3) are satisfied and the result follows by the asymptotic for $c(n)$ in BCMO. □

Asymptotic convexity

C.−Craig−Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. Assume that $H(1; q)$ and $H(\zeta; q)$ satisfy the conditions of BCMO. Then for sufficiently large n_1, n_2 we have

$$c(a, b; n_1)c(a, b; n_2) > c(a, b; n_1 + n_2).$$

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2. partition ranks congruent to $a \pmod b$ (Hou−Jagadeesan, Males)

Log-concavity

C.—Craig—Males, 2021

Let $0 \leq a < b$ and $b \geq 2$. Assume that $H(1; q)$ and $H(\zeta; q)$ satisfy the conditions of BCMO. For large enough n, we have

$$c(a, b; n)^2 \geq c(a, b; n-1)c(a, b; n+1).$$

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Ramanujan congruences, 1921

For $n \geq 0$ we have

$$p(5n+4) \equiv 0 \pmod{5},$$
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The ranks of the partitions of 4:

partition	rank
(4)	$3 \equiv 3 \pmod{5}$
$(3,1)$	$1 \equiv 1 \pmod{5}$
$(2,2)$	$0 \equiv 0 \pmod{5}$
$(2,1,1)$	$-1 \equiv 4 \pmod{5}$
$(1,1,1,1)$	$-3 \equiv 2 \pmod{5}$

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Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that

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The equidistribution of $N(a, b; n)$ was already proven by Males in 2021 using Ingham's Tauberian theorem.

The crank

$$\text{crank}(\lambda) := \begin{cases} \text{largest part of } \lambda & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0 \end{cases}$$

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$$(4),\ (\overline{4}),\ (3,1),\ (\overline{3},1),\ (3,\overline{1}),\ (\overline{3},\overline{1}),\ (2,2),\ (\overline{2},2),$$
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The *first residual crank* of an overpartition is given by the crank of the subpartition consisting of the non-overlined parts.

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Let $0 \leq a < b$ and $b \geq 2$. Then as $n \to \infty$ we have that

$$\overline{M}(a, b; n) = \frac{1}{8bn} e^{\pi\sqrt{n}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right).$$

Plane partitions

A *plane partition* of n is a two-dimensional array $\pi_{j,k}$ of non-negative integers $j, k \geq 1$, that is non-increasing in both variables, i.e., $\pi_{j,k} \geq \pi_{j+1,k}$, $\pi_{j,k} \geq \pi_{j,k+1}$ for all j and k, and fulfils $|\Lambda| := \sum_{j,k} \pi_{j,k} = n$.

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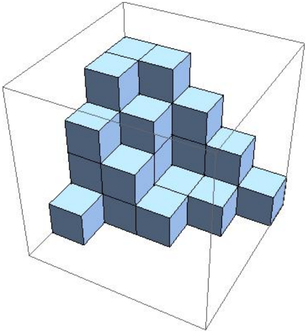
Thus we have $\mathrm{pp}(3) = 6$.

Plane partitions



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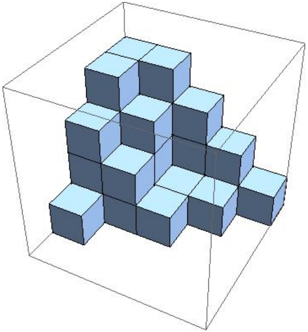

A plane partition may be represented visually by the placement of a stack of $\pi_{j,k}$ unit cubes above the point (j, k) in the plane, giving a three-dimensional solid.

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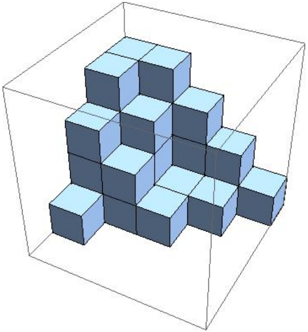

A plane partition may be represented visually by the placement of a stack of $\pi_{j,k}$ unit cubes above the point (j,k) in the plane, giving a three-dimensional solid.

The sum $|\Lambda|$ then describes the number of cubes of which the plane partition consists.

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$$\mathrm{pp}(a,b;n) \sim \frac{1}{b}\,\mathrm{pp}(n) \sim \frac{1}{b}\frac{\zeta(3)^{\frac{7}{56}}}{\sqrt{12\pi}}\left(\frac{n}{2}\right)^{-\frac{25}{36}} \exp\left(3\zeta(3)^{\frac{1}{3}}\left(\frac{n}{2}\right)^{\frac{2}{3}} + \zeta'(-1)\right).$$

Betti numbers of Hilbert schemes

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We define the Hilbert schemes

$$X_1 := \mathrm{Hilb}^{n,n+1,n+2}(0), \qquad X_2 := \mathrm{Hilb}^{n,n+2}(0),$$
$$X_3 := \mathrm{Hilb}^{n,n+2}\left(\mathbb{C}^2\right)_{\mathrm{tr}}, \qquad X_4 := \widehat{M}^m(c_N),$$

where $m \in \mathbb{N}$ and c_N is some prescribed homological data.

Betti numbers of Hilbert schemes

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Let $0 \leq a < b$ with $b \geq 2$ and

$$d(a,b) := \begin{cases} \frac{1}{b} & \text{if } b \text{ is odd,} \\ \frac{2}{b} & \text{if } a \text{ and } b \text{ are even,} \\ 0 & \text{if } a \text{ is odd and } b \text{ is even.} \end{cases}$$

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$$\frac{1}{2}B(a,b;X_1) \sim B(a,b;X_2) \sim B(a,b;X_3) = \frac{d(a,b)\sqrt{3}}{4\pi^2} e^{\pi\sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right)$$

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and

$$B(a,b;X_4) = \frac{d(a,b) n^{\frac{m-2}{2}}}{6^{\frac{1-m}{2}} 2\sqrt{2} c_m \pi^m} e^{\pi\sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

where $\prod_{j=1}^{m} \frac{1}{1-e^{-jz}} = \frac{1}{c_m z^m} + O(z^{-m+1})$.

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Proof for crank

Using orthogonality of roots of unity we have

$$\sum_{n\geq 0} M(a,b;n)q^n = \frac{1}{b}\sum_{n\geq 0} p(n)q^n + \frac{1}{b}\sum_{j=1}^{b-1} \zeta_b^{-aj} C\left(\zeta_b^j; q\right),$$

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with $(q;q)_\infty := \prod_{\ell=1}^{\infty}(1-q^\ell)$ and $F_1(\zeta;q) := \prod_{n=1}^{\infty}(1-\zeta q^n)$.

Proof for crank

As $z \to 0$ in D_θ, for $q = e^{-z}$ and ζ a primitive b-th root of unity (Bringmann–Craig–Males–Ono)

$$F_1\left(\zeta; e^{-z}\right) = \frac{1}{\sqrt{1-\zeta}}\, e^{-\frac{\zeta \Phi(\zeta,2,1)}{z}} \left(1 + O\left(|z|\right)\right),$$

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while on the minor arc, for some $\mathcal{C} > 0$

$$\left|\left(e^{-z}; e^{-z}\right)_\infty^{-1}\right| \leq x^{\frac{1}{2}} e^{\frac{\pi^2}{6x} - \frac{\mathcal{C}}{x}}.$$

Proof for crank

Using the definition of $F_1(\zeta; q)$

$$\left|\mathrm{Log}\left(\frac{1}{F_1(\zeta;q)}\right)\right| = \left|\sum_{k\geq 1}\frac{\zeta^k}{k}\frac{q^k}{1-q^k}\right|$$
$$\leq \left|\frac{\zeta q}{1-q}\right| - \frac{|q|}{1-|q|} + \log\left(\frac{1}{(|q|;|q|)_\infty}\right).$$

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$$\Rightarrow \qquad \left|\frac{1}{F_1(\zeta;q)}\right| \ll e^{-\frac{\mathcal{C}'}{x}}(|q|;|q|)_\infty^{-1},$$

for some $\mathcal{C}' > 0$.

Proof for crank

Since an analogous calculation holds for $F_1(\zeta^{-1}; q)$ one may conclude that

$$\left| C\left(\zeta_b^j; q\right)\right| < \left|(q;q)_\infty^{-1}\right|$$

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For the major arc

$$C\left(\zeta; q\right) \ll e^{-\frac{\pi^2}{6}\operatorname{Re}\left(\frac{1}{z}\right) + \operatorname{Re}\left(\frac{\zeta\Phi(\zeta,2,1)}{z}\right) + \operatorname{Re}\left(\frac{\zeta^{-1}\Phi(\zeta^{-1},2,1)}{z}\right)}.$$

Therefore

$$C\left(\zeta_b^j; q\right) = o\left((q;q)_\infty^{-1}\right)$$

if and only if

$$\left(\frac{\pi^2}{3} - \varepsilon - \phi_1 - \phi_1'\right)\frac{x}{|z|^2} > \left(\phi_2 + \phi_2'\right)\frac{y}{|z|^2},$$

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where $\phi_1 + i\phi_2 := \zeta_b^j \Phi(\zeta_b^j, 2, 1)$ and $\phi_1' + i\phi_2' := \zeta_b^{-j}\Phi(\zeta_b^{-j}, 2, 1)$.

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Note that $\phi_1 = \frac{\pi^2}{6} - \frac{\pi^2 j}{b}\left(1 - \frac{j}{b}\right) = \phi_1'$ and $\phi_2 = -\phi_2'$.

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Therefore, our assumption reduces to

$$\left(\frac{2\pi^2 j}{b}\left(1 - \frac{j}{b}\right) - \varepsilon\right)\frac{x}{|z|^2} > 0,$$

which holds, since we have $b > 0$, $1 \leq j \leq b-1$ and $x = \mathrm{Re}(z) > 0$. □

Proof for Betti numbers

Let X be a Hilbert scheme

$$G_X(T;q) := \sum_{n\geq 0} P(X;T)q^n,$$

with $P(X;T) := \sum_j b_j(X)T^j$ the *Poincaré polynomial*.

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Using orthogonality of roots of unity

$$\sum_{n\geq 0} B(a,b;X)q^n = \frac{1}{b}\sum_{r=0}^{b-1} \zeta_b^{-ar} G_X\left(\zeta_b^r;q\right).$$

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Boccalini's thesis states that

$$G_{X_1}(\zeta; q) = \sum_{n \geq 0} P\left(X_1; \zeta\right) q^n = \frac{1+\zeta^2}{(1-\zeta^2 q)(1-\zeta^4 q^2)} F_3\left(\zeta^2; q\right)^{-1},$$

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We obtain

$$\begin{aligned} H_{X_1}(a, b; q) &:= \sum_{n\geq 0} B(a, b; X_1) q^n \\ &= \frac{1}{b}\left(1+(-1)^a \delta_{2|b}\right) G_{X_1}(1; q) + \frac{1}{b} \sum_{\substack{0<r\leq b-1 \\ r\neq \frac{b}{2}}} \zeta_b^{-ar} G_{X_1}\left(\zeta_b^r; q\right). \end{aligned}$$

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Since

$$G_{X_1}(1; e^{-z}) = \frac{2}{(1-e^{-z})(1-e^{-2z})}(e^{-z}; e^{-z})_\infty^{-1}$$
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For $\zeta_b^r \neq 1$ it is enough to show that on the major and minor arcs,

$$G_{X_1}(\zeta_b^r; q) = o(G_{X_1}(1; q)).$$

Proof for Betti numbers

On the major arc (Bringmann–Craig–Males–Ono)

$$F_3(\zeta_b^{2r}; e^{-z})^{-1} \ll e^{\frac{\pi^2}{6z}} |z|^{-N},$$

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and therefore again $G_{X_1}(\zeta_b^r; q) = o(G_{X_1}(1; q))$.

Proof for Betti numbers

Thus toward $z = 0$ on the major arc we have

$$H_{X_1}(a, b; e^{-z}) = \frac{d(a, b)}{\sqrt{2\pi} z^{\frac{3}{2}}} e^{\frac{\pi^2}{6z}} (1 + O(|z|)).$$

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We are left to apply BCMO with $A = \frac{\pi^2}{6}, B = -\frac{3}{2}$, and $\alpha_0 = \frac{d(a,b)}{\sqrt{2\pi}}$ which yields that

$$B(a, b; X_1) = \frac{\sqrt{3} d(a, b)}{2\pi^2} e^{\pi \sqrt{\frac{2n}{3}}} \left(1 + O\left(n^{-\frac{1}{2}}\right)\right),$$

from which one may also conclude asymptotic equidistribution.

Proof for Betti numbers

Similarly, it is known that

$$G_{X_2}(\zeta;q) := \frac{1+\zeta^2-\zeta^2 q}{(1-\zeta^2 q)(1-\zeta^4 q^2)} F_3\left(\zeta^2;q\right)^{-1},$$

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An analogous argument to the case of X_1 holds. □

Thank you for your attention!